The cyclotomic trace map and values of zeta functions

Thomas Geisser

Abstract. We show that the cyclotomic trace map for smooth varieties over number rings can be interpreted as a regulator map and hence are related to special values of ζ-functions.

1. Introduction

The purpose of this paper is to show that the results of [**8**] can be used to relate the cyclotomic trace map from étale K-theory to topological cyclic homology

$$\mathrm{tr}_i : K_i^{\text{ét}}(X, \mathbb{Z}_p) \to \mathrm{TC}_i(X; p, \mathbb{Z}_p)$$

to arithmetic invariants if X is regular scheme, flat and proper over a number ring $\mathcal{O}_S$, and with good reduction at p. The main result of [**8**] implies that the map tr_i can be identified with the localization map

$$\mathrm{tr}_i : K_i^{\text{ét}}(X, \mathbb{Z}_p) \to K_i^{\text{ét}}(X \times_{\mathbb{Z}} \mathbb{Z}_p, \mathbb{Z}_p). \tag{1.1}$$

Both sides of (1.1) admit a hypercohomology spectral sequence of the form

$$E_2^{s,t} = H^s_{cont}(X, (\mathcal{K}/p^{\cdot})_{-t}) \Rightarrow K^{\text{ét}}_{-s-t}(X, \mathbb{Z}_p),$$

but the E_2-term is hard to control because the étale K-theory sheaf $(\mathcal{K}/p^r)_i$ is not known. In order to overcome this problem, we compare the map (1.1) to the map

$$K_i^{\text{ét}}(X \times_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p) \to K_i^{\text{ét}}(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p). \tag{1.2}$$

Using either the Lichtenbaum-Quillen conjecture or a result of Thomason, one can identify the maps (1.1) and (1.2) if one assumes that

1991 *Mathematics Subject Classification.* 19F27, 11R42, 11R70.

Key words and phrases.

Supported in part by NSF grant No. 0300133, and the Alfred P. Sloan Foundation.

$i > d = \dim X$ or $i \geq \frac{8}{3}(d+2)(d+3)(d+4) - 14$, respectively. Since p is invertible in (1.2), the étale K-theory sheaf can be identified, and the map (1.2) is the map on the abutments of the spectral sequences

$$\begin{array}{ccc} E_2^{s,t} = H^s(X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ E_2^{s,t} = H^s(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p). \end{array}$$

Thus we can relate the map (1.2) to maps between étale cohomology groups, which in turn are related to special values of L-functions.

In the second half of the paper we give concrete calculations in case X is the ring of integers $\operatorname{Spec} \mathcal{O}$ of a number field K. For $j = 1, 2$, the trace map tr_{2i-j} can be identified with the map

$$H^j(G_\Sigma, \mathbb{Z}_p(i)) \to \prod_{\mathfrak{p}|p} H^j(K_{\mathfrak{p}}, \mathbb{Z}_p(i)),$$

where G_Σ is Galois group of the maximal extension of K which is unramified outside of p and infinity. This map has been study in Iwasawa theory, and we translate results of Iwasawa theory into statements about the trace map. For example, we show that if $K/\mathbb{Q}$ is a totally real field, unramified at p, and $i \not\equiv 1 \mod p-1$ is an odd integer, then the trace map

$$\mathrm{tr}'_{2i-1} : K_{2i-1}(\mathcal{O}) \otimes \mathbb{Z}_p/\mathrm{tors} \to \mathrm{TC}_{2i-1}(\mathcal{O}, \mathbb{Z}_p).$$

is a map between free $\mathbb{Z}_p$-modules of rank $d = [K : \mathbb{Q}]$. Its cokernel is finite if and only if a conjecture of Schneider holds, and in this case the order of the cokernel is related to the p-adic L-function as follows:

$$|H^2(G_\Sigma, \mathbb{Z}_p(i))| \cdot |\operatorname{coker} \mathrm{tr}'_{2i-1}| = |L_p(K, \omega^{1-i}, i)|_p^{-1}.$$

Convention: All cohomology groups are étale cohomology in case of schemes, and Galois cohomology groups in case of fields.

2. Preliminaries

We recall some facts on algebraic K-theory and topological cyclic homology, see [**6**], [**7**] and [**11**].

2.1. K-theory. For every henselian pair (A, I) such that m is invertible in A, and for all $i \geq 0$, we have the isomorphism of Gabber [**4**] and Suslin [**26**]

$$K_i(A, \mathbb{Z}/m) \xrightarrow{\sim} K_i(A/I, \mathbb{Z}/m).$$

Together with the calculation of the K-theory of an algebraically closed field [**26**] by Suslin, this implies that on every scheme X such that m

is invertible on X, the K-theory sheaf with coefficients for the étale topology can be identified as follows

$$(\mathcal{K}/m)_n = \begin{cases} \mu_m^{\otimes \frac{n}{2}} & n \geq 0 \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \tag{2.1}$$

Let R be a local ring, such that (R, pR) is a henselian pair, and such that p is not a zero divisor. Then [**8**] the reduction map

$$K_i(R, \mathbb{Z}/p^r) \to \{K_i(R/p^s, \mathbb{Z}/p^r)\}_s \tag{2.2}$$

is an isomorphism of pro-abelian groups. This generalizes the result of Suslin and Panin [**26, 20**] for R a henselian valuation ring of mixed characteristic $(0, p)$.

For a presheaf of spectra $\mathcal{F}$ on a site X_τ, and a covering $\mathcal{U} = \{U_i\}$ of X, Thomason [**28**, Def. 1.9, 1.33] defines the Čech hypercohomology spectrum $\check{\mathbb{H}}^{\cdot}(\mathcal{U}, \mathcal{F})$ and the sheaf hypercohomology spectrum $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$. There are natural augmentation maps $\tau : \mathcal{F}(X) \to \check{\mathbb{H}}^{\cdot}(\mathcal{U}, \mathcal{F})$ and $\eta : \mathcal{F}(X) \to \mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$. If τ is the Zariski or Nisnevich topology on a noetherian scheme X of finite Krull dimension, then it is a theorem of Brown-Gersten [**2**] and Nisnvevich [**19**], respectively, that the augmentation map $\eta : K(X) \to \mathbb{H}^{\cdot}(X_\tau, K)$ is a homotopy equivalence. Moreover, the Čech hypercohomology of the sheaf hypercohomology agrees with the sheaf hypercohomololgy [**28**, Cor. 1.47]

$$\mathbb{H}^{\cdot}(X_{\mathrm{Zar}}, \mathcal{F}) \cong \check{\mathbb{H}}^{\cdot}(\mathcal{U}, \mathbb{H}^{\cdot}(-, \mathcal{F})). \tag{2.3}$$

One important feature of $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F})$ is that it comes equipped with a spectral sequence [**28**, Prop. 1.36]

$$E_2^{s,t} = H^s(X_\tau, \tilde{\pi}_{-t}\mathcal{F}) \Rightarrow \pi_{-s-t}\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}), \tag{2.4}$$

where $\tilde{\pi}_i\mathcal{F}$ is the sheaf associated to the presheaf of homotopy groups $U \mapsto \pi_i\mathcal{F}(U)$. If X_τ has finite cohomological dimension, then the spectral sequence converges.

For a pro-presheaf $\mathcal{F}^{\cdot}$ of spectra on X_τ, one defines the hypercohomology spectrum $\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^{\cdot}) := \operatorname{holim}_r \mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^r)$. For a (complex of) sheaves of abelian groups $A^{\cdot}$ on X_τ, Jannsen [**14**] defines continuous cohomology groups $H^j_{cont}(X_\tau, A^{\cdot})$ as the derived functors of the functor $A^{\cdot} \mapsto \lim_r \Gamma(X, A^r)$, and we get a spectral sequence [**7**]

$$E_2^{s,t} = H^s_{cont}(X_\tau, \tilde{\pi}_{-t}\mathcal{F}^{\cdot}) \Rightarrow \pi_{-s-t}\mathbb{H}^{\cdot}(X_\tau, \mathcal{F}^{\cdot}). \tag{2.5}$$

If $X_{\text{ét}}$ is the small étale site of the scheme X, then we write $K_i^{\text{ét}}(X, \mathbb{Z}_p)$ for the homotopy groups $\pi_i \operatorname{holim}_r \mathbb{H}^{\cdot}(X_{\text{ét}}, K/p^r)$. If p is invertible on X,

we write $H^i(X,\mathbb{Z}_p(n))$ for $H^i_{cont}(X_{\text{ét}},\mu_{p^\cdot}^{\otimes n})$. In view of (2.1) the spectral sequence (2.5) takes the form

$$E_2^{s,t} = H^s(X,\mathbb{Z}_p(-\tfrac{t}{2})) \Rightarrow K^{\text{ét}}_{-s-t}(X,\mathbb{Z}_p). \tag{2.6}$$

The Lichtenbaum-Quillen conjecture states that for i greater than the cohomological dimension of X, the canonical map from K-theory to étale K-theory

$$K_i(X,\mathbb{Z}_p) \to K_i^{\text{ét}}(X,\mathbb{Z}_p)$$

is an isomorphism. The Lichtenbaum-Quillen conjecture is a consequence of the Beilinson-Lichtenbaum conjecture, whose proof has been announced by Voevodsky [**30**]. The following special case is known by Hesselholt and Madsen [**13**, Thm. A]:

THEOREM 2.1. *Let K be a complete discrete valuation field of characteristic* 0 *with perfect residue field of characteristic $p > 2$. Then for $i \geq 1$,*

$$K_i(K,\mathbb{Z}/p^r) \cong K_i^{\text{ét}}(K,\mathbb{Z}/p^r).$$

This has been generalized to certain discrete valuation rings with non-perfect residue fields in [**9**]. See [**11**] for a survey of these results.

2.2. Topological cyclic homology. Using the hyper-cohomology construction of Thomason, one can [**7**] extend the definition of topological Hochschild homology $\mathrm{TH}(A)$ for a ring A by considering the presheaf of spectra $\mathrm{TH} : U \mapsto \mathrm{TH}(\Gamma(U,\mathcal{O}_U))$, and setting

$$\mathrm{TH}(X_\tau) = \mathbb{H}^{\cdot}(X_\tau,\mathrm{TH}). \tag{2.7}$$

PROPOSITION 2.2. *a)* [**7**, Cor. 3.3.3] *If the Grothendieck topology τ on the scheme X is coarser than or equal to the étale topology, then $\mathrm{TH}(X_\tau)$ is independent of the topology (and we drop τ from the notation).*

b) [**7**, Cor. 3.2.2] *If X is the spectrum of a ring A, then* $\mathrm{TH}(A) \xrightarrow{\sim} \mathrm{TH}(X)$.

It follows from the proposition and (2.3) that for a notherian scheme of finite Krull dimension,

$$\mathrm{TH}(X) \cong \mathrm{TH}(X_{\mathrm{Zar}}) \cong \check{\mathbb{H}}^{\cdot}(\mathcal{U},\mathrm{TH}). \tag{2.8}$$

In particular, $\mathrm{TH}(X)$ is determined by the spectra $\mathrm{TH}(U_i)$ for $U_i \in \{\mathcal{U}\}$.

The spectrum $\mathrm{TR}^m(X;p)$ is the fixed point spectrum under of the cyclic subgroup of roots of unity $\mu_{p^{m-1}} \subseteq S^1$ acting on $\mathrm{TH}(X)$; let $\mathrm{TR}^m(X;p,\mathbb{Z}/p^r)$ be the version with coefficients. There are natural maps called Frobenius and restriction map

$$F,R : \mathrm{TR}^m(X;p,\mathbb{Z}/p^r) \to \mathrm{TR}^{m-1}(X;p,\mathbb{Z}/p^r),$$

and topological cyclic homology $\mathrm{TC}^m(X;p,\mathbb{Z}/p^r)$ is the homotopy equalizer of F and R. We view $\mathrm{TC}^{\cdot}(X;p,\mathbb{Z}/p^r)$ as a pro-spectrum with R as the structure map, and define

$$\mathrm{TC}(X;p,\mathbb{Z}_p) = \operatorname*{holim}_{m,r} \mathrm{TC}^m(X;p,\mathbb{Z}/p^r).$$

If $(TC^m/p^r)_i$ is the sheaf associated to the presheaf $U \mapsto \mathrm{TC}_i^m(U;p,\mathbb{Z}/p^r)$, then (2.5) takes the form

$$E_2^{s,t} = H^s_{cont}(X_{\text{ét}}, (TC^{\cdot}/p^{\cdot})_{-t}) \Rightarrow \mathrm{TC}_{-s-t}(X;p,\mathbb{Z}_p). \tag{2.9}$$

If we use the Zariski or Nisnevich topology instead of the étale topology, we get a different spectral sequence with the same abutment. The statements of Proposition 2.2 and (2.8) procreate to analog statements for TC.

Topological cyclic homology comes equipped with the cyclotomic trace map

$$\mathrm{tr}' : K(X,\mathbb{Z}_p) \to \mathrm{TC}(X;p,\mathbb{Z}_p).$$

In [**12**], Hesselholt and Madsen show that the trace map is an isomorphism in non-negative degrees for a finite algebra over the Witt ring of a perfect field. Since Thomason's construction is functorial, this factors by Proposition 2.2 a) through

$$\mathrm{tr} : K^{\text{ét}}(X,\mathbb{Z}_p) \to \mathrm{TC}(X_{\text{ét}};p,\mathbb{Z}_p) \cong \mathrm{TC}(X;p,\mathbb{Z}_p).$$

Theorem 2.3. [**8**, Thm. A] *Let X is a smooth, proper scheme over a henselian discrete valuation ring V of mixed characteristic $(0,p)$. Then the cyclotomic trace map from étale K-theory to topological cyclic homology*

$$K_i^{\text{ét}}(X,\mathbb{Z}_p) \xrightarrow{\mathrm{tr}_i} \mathrm{TC}_i(X;p,\mathbb{Z}_p)$$

is an isomorphism.

3. The trace map for arithmetic schemes

We fix a prime $p \neq 2$, let $\mathbb{Z}_h$ the henselization, and $\mathbb{Z}_p$ the completion of the integers at p, $\mathbb{Q}_h = \mathbb{Z}_h[\frac{1}{p}]$, and $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$. We fix a number field K, and let $\mathcal{O}$ be its ring of integers. For a set of prime ideals S of $\mathcal{O}$ not containing any of the primes dividing p, we let $\mathcal{O}_S$ be the S-integers of K.

Proposition 3.1. *Let X be a regular scheme, flat and proper over* $\operatorname{Spec}\mathcal{O}_S$*, with good reduction at p. Then there is a commutative diagram*

$$\begin{array}{ccccc} K_i(X,\mathbb{Z}_p) & \xrightarrow{\alpha_i} & K_i^{\text{ét}}(X,\mathbb{Z}_p) & \xrightarrow{f_i} & K_i^{\text{ét}}(X\times_{\mathbb{Z}}\mathbb{Z}_h,\mathbb{Z}_p) \\ \downarrow{\scriptstyle \mathrm{tr}_i'} & & {\scriptstyle \mathrm{tr}_i}\downarrow & & {\scriptstyle\cong}\downarrow \\ \mathrm{TC}_i(X;p,\mathbb{Z}_p) & = & \mathrm{TC}_i(X;p,\mathbb{Z}_p) & \xrightarrow{\sim} & \mathrm{TC}_i(X\times_{\mathbb{Z}}\mathbb{Z}_h;p,\mathbb{Z}_p). \end{array}$$

In particular, the cyclotomic trace map is isomorphic to the composition

$$K_i(X,\mathbb{Z}_p) \xrightarrow{\alpha} K_i^{\text{ét}}(X,\mathbb{Z}_p) \xrightarrow{f_i} K_i^{\text{ét}}(X\times\mathbb{Z}_h,\mathbb{Z}_p).$$

The same statements hold with $\mathbb{Z}_p$ instead of $\mathbb{Z}_h$.

Proof. Clearly the diagram commutes, and the right vertical map is an isomorphism by Theorem 2.3. To show the isomorphism

$$\mathrm{TC}_i(X;p,\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{TC}_i(X\times_{\mathbb{Z}}\mathbb{Z}_h;p,\mathbb{Z}_p),$$

let $\mathcal{U}=\{U_i\}$ be an affine open covering of X. Then $\mathcal{U}_h=\{U_i\times_{\mathbb{Z}}\mathbb{Z}_h\}$ is an affine open covering of $X\times_{\mathbb{Z}}\mathbb{Z}_h$. By (2.8), it suffices to show that $\mathrm{TC}(U_i;p,\mathbb{Z}_p)$ is homotopy equivalent to $\mathrm{TC}(U_i\times_{\mathbb{Z}}\mathbb{Z}_h;p,\mathbb{Z}_p)$. Thus we can assume that $X=\operatorname{Spec}R$, with R flat and of finite type over $\mathcal{O}_S$. Then p is not a zero divisor in R, and the rings R/p^s and $(R\otimes\mathbb{Z}_h)/p^s$ are isomorphic. By [**8**, Addendum 3.1.2] we get $\mathrm{TC}_i(R;p,\mathbb{Z}_p)\xrightarrow{\sim}\mathrm{TC}_i(R\otimes\mathbb{Z}_h;p,\mathbb{Z}_p)\xrightarrow{\sim}\mathrm{TC}_i(R\otimes\mathbb{Z}_p;p,\mathbb{Z}_p)$. □

The problem in evaluating the trace map tr_i is the calculation of the étale K-theory groups involved. We solve this problem by localizing away from p:

Theorem 3.2. *Let X be regular scheme, flat and proper over* $\operatorname{Spec}\mathcal{O}_S$*, with good reduction at p.*

a) If $K_i^{\text{ét}}(X\times_{\mathbb{Z}}\mathbb{Z}[\frac{1}{p}],\mathbb{Z}_p)\xrightarrow{g_i}K_i^{\text{ét}}(X\times_{\mathbb{Z}}\mathbb{Q}_p,\mathbb{Z}_p)$ is the restriction map, then there is an exact sequence

$$0\to\operatorname{coker}\mathrm{tr}_i\to\operatorname{coker}g_i\xrightarrow{\delta}\ker\mathrm{tr}_{i-1}\to\ker g_{i-1}\to 0.$$

b) Let d be the relative dimension of X over $\mathcal{O}_S$. If $i\geq\frac{8}{3}(d+2)(d+3)(d+4)-14$, or if the Lichtenbaum-Quillen conjecture holds and $i>d+1$, then the map δ is the zero map.

Proof. a) For a closed subset Z of a scheme X with open complement U, we let $K^{\text{ét},Z}(X,\mathbb{Z}_p)$ be the homotopy fiber of the natural map $K^{\text{ét}}(X,\mathbb{Z}_p)\to K^{\text{ét}}(U,\mathbb{Z}_p)$. The closed complement $Y=X\times_{\mathbb{Z}}\mathbb{F}_p$ of

$X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ in X is isomorphic to the closed complement of $X \times \mathbb{Q}_h$ in $X \times \mathbb{Z}_h$. Consider the natural map of long exact sequences

(3.1)

$$\begin{array}{ccccc} K_i^{\text{ét},Y}(X,\mathbb{Z}_p) & \longrightarrow & K_i^{\text{ét}}(X,\mathbb{Z}_p) & \longrightarrow & K_i^{\text{ét}}(X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}],\mathbb{Z}_p) \\ \downarrow & & {\scriptstyle f_i}\downarrow & & {\scriptstyle g_i^h}\downarrow \\ K_i^{\text{ét},Y}(X \times \mathbb{Z}_h,\mathbb{Z}_p) & \longrightarrow & K_i^{\text{ét}}(X \times \mathbb{Z}_h,\mathbb{Z}_p) & \xrightarrow{j^*} & K_i^{\text{ét}}(X \times \mathbb{Q}_h,\mathbb{Z}_p). \end{array}$$

By Theorem 2.3, f_i can be identified with tr_i, and by the following Lemma g_i can be identified with g_i^h. According to [**29**, Thm. D.4], there are spectral sequences

$$\begin{array}{ccc} E_2^{s,t} = H_Y^s(X,(\mathcal{K}/p^r)_{-t}) & \Rightarrow & K_{-s-t}^{\text{ét},Y}(X,\mathbb{Z}/p^r) \\ \downarrow & & \downarrow \\ E_2^{s,t} = H_Y^s(X \times \mathbb{Z}_h,(\mathcal{K}/p^r)_{-t}) & \Rightarrow & K_{-s-t}^{\text{ét},Y}(X \times \mathbb{Z}_h,\mathbb{Z}/p^r). \end{array}$$

By étale excision [**18**, Prop. 1.27], the E_2-terms of the two spectral sequences are isomorphic, because $X \times_{\mathbb{Z}} \mathbb{Z}_h$ is the direct limit of étale neighborhoods of Y in X. Taking the limit over r shows that the two left terms in diagram (3.1) are isomorphic, and we get a) by an easy diagram chase.

b) It suffices to show that the map j^* in diagram (3.1) is surjective. Consider the commutative diagram

$$\begin{array}{ccc} K_i(X \times \mathbb{Z}_h,\mathbb{Z}_p) & \longrightarrow & K_i(X \times \mathbb{Q}_h,\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ K_i^{\text{ét}}(X \times \mathbb{Z}_h,\mathbb{Z}_p) & \longrightarrow & K_i^{\text{ét}}(X \times \mathbb{Q}_h,\mathbb{Z}_p). \end{array}$$

The right hand map is surjective for $i \geq \frac{8}{3}(d+2)(d+3)(d+4) - 14$ by Thomason [**27**], and the Lichtenbaum-Quillen conjecture implies that the right hand map is an isomorphism for $i > d+1$. On the other hand, by localization, the cokernel of the upper map is contained in $K_{i-1}(Y,\mathbb{Z}_p)$, which is zero for $i-1 > d$ by [**10**] because Y is smooth. Hence the lower map is surjective □

LEMMA 3.3. *Let X be a smooth scheme over* $\operatorname{Spec}\mathbb{Q}_h$*. Then for any i, we have*

$$K_i^{\text{ét}}(X,\mathbb{Z}_p) \cong K_i^{\text{ét}}(X \times_{\mathbb{Q}_h} \mathbb{Q}_p,\mathbb{Z}_p).$$

PROOF. In view of spectral sequence (2.6),

$$\begin{array}{ccc} E_2^{s,t} = H^s(X, \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X, \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ E_2^{s,t} = H^s(X \times \mathbb{Q}_p, \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X \times_{\mathbb{Q}_h} \mathbb{Q}_p, \mathbb{Z}_p), \end{array}$$

it suffices to show that the canonical map induces an isomorphism on E_2-terms. By [**14**, Cor. 3.4] there are spectral sequences of étale cohomology groups

$$\begin{array}{ccc} E_2^{s,t} = H^a(\mathbb{Q}_h, H^b(X \times \bar{\mathbb{Q}}_h, \mathbb{Z}_p(n))) & \Rightarrow & H^{a+b}(X, \mathbb{Z}_p(n)) \\ \downarrow & & \downarrow \\ E_2^{s,t} = H^a(\mathbb{Q}_p, H^b(X \times \bar{\mathbb{Q}}_p, \mathbb{Z}_p(n))) & \Rightarrow & H^{a+b}(X \times \mathbb{Q}_p, \mathbb{Z}_p(n)). \end{array}$$

The Galois groups of $\mathbb{Q}_h$ and $\mathbb{Q}_p$ are isomorphic, and so are the Galois modules. Indeed, this is a consequence of the smooth base change theorem for finite coefficients [**18**, Cor. VI 4.3], and immediately extends to continuous cohomology. □

COROLLARY 3.4. *Let X be regular scheme, flat and proper over $\operatorname{Spec} \mathcal{O}_S$ with good reduction at the primes above p. Assume that the Lichtenbaum-Quillen conjecture holds, that $i > \dim X$ and $p > \dim X + 2$. Then the map $\operatorname{tr}_i : K_i^{\text{ét}}(X, \mathbb{Z}_p) \to \operatorname{TC}_i(X, \mathbb{Z}_p)$ is isomorphic to the sum of localization maps*

$$\bigoplus_a H^{2a-i}(X \times_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{p}], \mathbb{Z}_p(a)) \to \bigoplus_a H^{2a-i}(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p(a)).$$

PROOF. The spectral sequences

$$\begin{array}{ccc} E_2^{s,t} = H^s(X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ E_2^{s,t} = H^s(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p(-\frac{t}{2})) & \Rightarrow & K^{\text{ét}}_{-s-t}(X \times_{\mathbb{Z}} \mathbb{Q}_p, \mathbb{Z}_p) \end{array}$$

degenerate at E_2 with split filtration for $p > \frac{cd_p X}{2}$ by Soulé [**24**, Thm. 1]. □

The localization map for étale cohomology is related to L-functions and p-adic L-functions by Iwasawa theory. In the following sections, we give examples for number fields, in particular totally real number fields. It should be possible to extend these results to Dirichlet characters, elliptic curves with complex multiplication, or Hecke characters of imaginary quadratic fields as in [**25, 5**].

4. Number fields

In the case of a number field, we can make the calculations of the last section more explicit. The translation of results of Iwasawa-theory into results on regulators are similar to [**15**]. We keep the notation of the previous section. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_S$ dividing $p \neq 2$, $\mathcal{O}_\mathfrak{p}$ be the completion of $\mathcal{O}_S$ at $\mathfrak{p}$, $K_\mathfrak{p}$ its quotient field, and $k_\mathfrak{p} \cong \mathcal{O}_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ its residue field. Similarly, let $\mathcal{O}^h_\mathfrak{p}$ be the henselization of $\mathcal{O}_S$ at $\mathfrak{p}$, and $K^h_\mathfrak{p}$ its quotient field. The residue fields of $\mathcal{O}^h_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ are canonicallly isomorphic. Note that $K \otimes_\mathbb{Z} \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} K_\mathfrak{p}$, $\mathcal{O}_S \otimes_\mathbb{Z} \mathbb{Z}_p \cong \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}$, and $(\mathcal{O}_S \otimes_\mathbb{Z} \mathbb{F}_p)^{red} \cong \prod_{\mathfrak{p}|p} k_\mathfrak{p}$, and similarly for the henselization.

Proposition 4.1. *For $i > 1$ and $j = 1, 2$ we have the following isomorphisms*

$$K^{\text{ét}}_{2i-j}(\mathcal{O}_S, \mathbb{Z}_p) \xrightarrow{\sim} K^{\text{ét}}_{2i-j}(\mathcal{O}_S[\tfrac{1}{p}], \mathbb{Z}_p) \xrightarrow{\sim} H^j(\operatorname{Spec} \mathcal{O}_S[\tfrac{1}{p}], \mathbb{Z}_p(i))$$
$$K^{\text{ét}}_{2i-j}(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) \xrightarrow{\sim} K^{\text{ét}}_{2i-j}(K_\mathfrak{p}, \mathbb{Z}_p) \xrightarrow{\sim} H^j(\operatorname{Spec} K_\mathfrak{p}, \mathbb{Z}_p(i)).$$

Proof. Since $\operatorname{Spec} \mathcal{O}_S[\frac{1}{p}]$ and $K_\mathfrak{p}$ have cohomological dimension 2 if $p \neq 2$, the right hand isomorphism follows from the spectral sequence (2.6) and $H^0(\operatorname{Spec} \mathcal{O}_S[\frac{1}{p}], \mathbb{Z}_p(i)) = H^0(\operatorname{Spec} K_\mathfrak{p}, \mathbb{Z}_p(i)) = 0$ for $i > 0$.

Consider the commutative diagram of long exact sequences

$$\begin{array}{ccccc}
K_i(k_\mathfrak{p}, \mathbb{Z}_p) & \longrightarrow & K_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) & \longrightarrow & K_i(K_\mathfrak{p}, \mathbb{Z}_p) \\
\downarrow & & \downarrow & & \downarrow \\
K^{\text{ét}, k_p}_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) & \longrightarrow & K^{\text{ét}}_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) & \longrightarrow & K^{\text{ét}}_i(K_\mathfrak{p}, \mathbb{Z}_p)
\end{array}$$

The middle map is an isomorphism by [**12**] and the the right map is an isomorphism by Theorem 2.1, hence in view of $K_i(k_\mathfrak{p}, \mathbb{Z}_p) = 0$ for $i > 0$ we get the local result together with $K^{\text{ét}, k_p}_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) = 0$ for $i > 0$. Comparing the lower row with the analog row for the henselization, we get $K^{\text{ét}, k_\mathfrak{p}}_i(\mathcal{O}^h_\mathfrak{p}, \mathbb{Z}_p) = 0$ for $i > 0$. Indeed, $K^{\text{ét}}_i(K^h_\mathfrak{p}, \mathbb{Z}_p) \cong K^{\text{ét}}_i(K_\mathfrak{p}, \mathbb{Z}_p)$ by Lemma 3.3, and by (2.2) there are isomorphisms

$$\begin{array}{ccc}
K_i(\mathcal{O}^h_\mathfrak{p}, \mathbb{Z}_p) & \xrightarrow{\sim} & K_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p) \\
\cong\downarrow & & \cong\downarrow \\
K^{\text{ét}}_i(\mathcal{O}^h_\mathfrak{p}, \mathbb{Z}_p) & \longrightarrow & K^{\text{ét}}_i(\mathcal{O}_\mathfrak{p}, \mathbb{Z}_p),
\end{array}$$

Using étale excission as in the proof of Theorem 3.2 a), we see that $K^{\text{ét}, k_\mathfrak{p}}_i(\mathcal{O}_S, \mathbb{Z}_p) \cong K^{\text{ét}, k_\mathfrak{p}}_i(\mathcal{O}^h_\mathfrak{p}, \mathbb{Z}_p) = 0$, hence the global result. □

Let Σ be the set of primes of $\mathcal{O}$ dividing p or infinity, and G_Σ be the Galois group of the maximal extension of K unramified outside Σ.

Then by [**3**, §3.2],

$$H^n(\operatorname{Spec}\mathcal{O}[\tfrac{1}{p}], \mathbb{Z}_p(i)) \cong H^n_{cont}(G_\Sigma, \mathbb{Z}_p(i)),$$

where the left hand side is étale cohomology and the right hand side is continuous Galois cohomology. Proposition 4.1 for $S = \emptyset$ shows that for $i > 0$ the trace map can be identified:

$$\begin{array}{ccc} K^{\text{ét}}_{2i-j}(\mathcal{O}, \mathbb{Z}_p) & \xrightarrow{tr_{2i-j}} & \mathrm{TC}_{2i-j}(\mathcal{O}; p, \mathbb{Z}_p) \\ \| & & \| \\ H^j(G_\Sigma, \mathbb{Z}_p(i)) & \longrightarrow & \prod_{\mathfrak{p}|p} H^j(K_\mathfrak{p}, \mathbb{Z}_p(i)). \end{array}$$

The following fundamental conjecture is due to Schneider [**21**, p. 129]:

Conjecture S(K,i) *The group $H^2(G_\Sigma, \mathbb{Z}_p(i))$ is torsion.*.

By Soulé [**22**], $S(K, i)$ holds for all $i > 1$, and $S(K, i)$ is by the long exact coefficient sequence equivalent to $H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$.

LEMMA 4.2. *Let K be a number field of degree d over $\mathbb{Q}$ with r_1 real and r_2 complex embeddings.*

a) [**21**, Satz 3.2, 3.4] *If $i \neq 0, 1$, then* $\operatorname{rank}_{\mathbb{Z}_p} \prod_{\mathfrak{p}|p} H^1(K_\mathfrak{p}, \mathbb{Z}_p(i)) = d$, *and the group $H^2(K_\mathfrak{p}, \mathbb{Z}_p(i))$ is finite.*

b) [**21**, Satz 4.6]

$$\operatorname{rank} H^1(G_\Sigma, \mathbb{Z}_p(i)) = \begin{cases} r_2 + \operatorname{rank} H^2(G_\Sigma, \mathbb{Z}_p(i)) & i \neq 0 \text{ even;} \\ r_1 + r_2 + \operatorname{rank} H^2(G_\Sigma, \mathbb{Z}_p(i)) & i \neq 1 \text{ odd.} \end{cases}$$

For $i > 0$, the Tate-Poitou exact sequence [**17**]

$$\begin{aligned} (4.1)\quad & 0 \to H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to \\ & H^1(G_\Sigma, \mathbb{Z}_p(i)) \xrightarrow{\mathrm{tr}_{2i-1}} \prod_{\mathfrak{p}|p} H^1(K_\mathfrak{p}, \mathbb{Z}_p(i)) \to H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to \\ & H^2(G_\Sigma, \mathbb{Z}_p(i)) \xrightarrow{\mathrm{tr}_{2i-2}} \prod_{\mathfrak{p}|p} H^2(K_\mathfrak{p}, \mathbb{Z}_p(i)) \to H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to 0. \end{aligned}$$

can be used to get results on the trace map.

For example, $S(K, 1-i)$ is equivalent to the injectivity of tr_{2i-1}, and the kernels of cokernels of the trace maps can be expressed in terms of

cohomology groups:

$$\ker \mathrm{tr}_{2i-1} \cong H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \tag{4.2}$$

$$0 \to \operatorname{coker} \mathrm{tr}_{2i-1} \to H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \to \ker \mathrm{tr}_{2i-2} \to 0 \tag{4.3}$$

$$\operatorname{coker} \mathrm{tr}_{2i-2} \cong H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^* \tag{4.4}$$

Moreover, by [**21**, Satz 5 ii]

$$\begin{aligned}\ker tr_{2i-2} &= \operatorname{coker}\left(H^1(K_p, \mathbb{Z}_p(i)) \to H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*\right)\\ &= \ker\left(H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \to \prod_{\mathfrak{p}|p} H^1(K_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^*\right)\\ &= \frac{\operatorname{div} H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(i))}{\operatorname{Div} H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(i))}.\end{aligned}$$

4.1. Totally real fields. Let K be a totally real field of degree d over $\mathbb{Q}$. Let w_i and $w_{\mathfrak{p},i}$ be the order of the group $H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(j))$ and $H^0(K_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p(j))$, respectively. We normalize the p-adic absolute value such that $|p|_p = \frac{1}{p}$, and write equalities of p-powers so that only positive powers appear on both sides of an equation.

Lemma 4.3. *a) Let $i > 0$ be even. Then the groups $H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong H^1(G_\Sigma, \mathbb{Z}_p(i))$, and $H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong H^2(G_\Sigma, \mathbb{Z}_p(i))$ are finite.*

b) Let $i \neq 1$ be odd. Then $H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$, $H^1(G_\Sigma, \mathbb{Z}_p(i)) \cong \mathbb{Z}_p^d$, and $S(K, i)$ holds.

c) If $K/\mathbb{Q}$ is unramified at p and $p - 1 \not| i$, then for every $\mathfrak{p}|p$,

$$H^0(K_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^2(K_\mathfrak{p}, \mathbb{Z}_p(1-i)) = 0.$$

Proof. a) By $S(K, i)$ for $i > 1$ and Lemma 4.2 b), we get that $H^1(G_\Sigma, \mathbb{Q}_p(i)) = H^2(G_\Sigma, \mathbb{Q}_p(i) = 0$ for $i > 0$ even. The result now follows from the long exact coefficient sequence.

b) Since K is totally real and $p \neq 2$, the degree of the extension $K(\mu_p)/K$ is divisible by two, hence $|H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i))| = \max\{p^j : [K(\mu_{p^j}) : K]|i\} = 1$. As this group is isomorphic to the torsion subgroup of $H^1(G_\Sigma, \mathbb{Z}_p(i))$, the latter group is torsion free. The group $H^2(G_\Sigma, \mathbb{Z}_p(i))$ is finite because $H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is zero. For $i > 1$ this follows from $S(K, i)$, and for $i < 0$ its dual is a subgroup of $H^1(G_\Sigma, \mathbb{Z}_p(1-i))$ by (4.1), and the latter is finite by a).

c) Since the extension $K_\mathfrak{p}/\mathbb{Q}_p$ is unramified by hypothesis. But $\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p$ is totally ramified at p and has degree $p-1$, hence the same holds for the extension $K_\mathfrak{p}(\mu_p)/K_\mathfrak{p}$. We can now use the argument of b) together with local duality [**21**, Satz 2.4] ∎

Proposition 4.4. *Let $i>0$ be even. Then* $\ker \mathrm{tr}_{2i-1} = \operatorname{coker} \mathrm{tr}_{2i-2} = 0$*, and*

$$|\zeta_K(1-i)|_p^{-1} \cdot |\ker \mathrm{tr}_{2i-2}| \cdot \prod_{\mathfrak{p}} w_{\mathfrak{p},1-i} = w_i.$$

Proof. The first two statements follow from Lemma 4.3 b), and (4.2). For the zeta value, we have

$$|\zeta_K(1-i)|_p = \frac{|H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i))|}{|H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i))|} = \frac{|H^1(G_\Sigma, \mathbb{Z}_p(i))|}{|H^2(G_\Sigma, \mathbb{Z}_p(i))|}$$
$$= \frac{|\ker \mathrm{tr}_{2i-2}| \cdot \prod_{\mathfrak{p}|p} |H^2(K_\mathfrak{p}, \mathbb{Z}_p(i))|}{w_i} = \frac{|\ker \mathrm{tr}_{2i-2}| \cdot \prod_{\mathfrak{p}} w_{\mathfrak{p},1-i}}{w_i}.$$

The first equality is [**1**, Thm. 6.2], the second equality is Lemma 4.3 a), the third equality follows from (4.1) because tr_{2i-2} is surjective, and the last equality is local duality. □

Theorem 4.5. *Let $i>0$ is odd. Then $S(K,1-i)$ holds if and only if* $\mathrm{tr}_{2i-1}=0$ *if and only if* tr_{2i-1} *has finite cokernel. In this case,*

$$|\operatorname{coker} \mathrm{tr}_{2i-1}| \cdot |H^2(G_\Sigma, \mathbb{Z}_p(i))| = \prod_{\mathfrak{p}|p} w_{\mathfrak{p},1-i} \cdot |L_p(K,\omega^{1-i},i)|_p^{-1}.$$

Proof. If $i>1$ is odd, then by Lemma 4.3 b)

$$\operatorname{rank} H^1(G_\Sigma, \mathbb{Z}_p(i)) = \operatorname{rank} \prod_{\mathfrak{p}|p} H^1(K_\mathfrak{p}, \mathbb{Z}_p(i)) = d.$$

By the Tate-Poitou sequence (4.1), $H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = 0$ if and only if $\ker \mathrm{tr}_{2i-1} = 0$ if and only if tr_{2i-1} has finite cokernel. In this case, by [**1**, Thm. 6.1], and (4.1),

$$|L_p(K,\omega^{1-i},i)|_p = \frac{|H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|}{|H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(1-i))|}$$
$$= \frac{\prod_{\mathfrak{p}|p} |H^2(K_\mathfrak{p}, \mathbb{Z}_p(i))|}{|\operatorname{coker} \mathrm{tr}_{2i-1}| \cdot |H^2(G_\Sigma, \mathbb{Z}_p(i))|}.$$

The result follows with local duality. □

The first result in this direction is due to Soulé [**23**]. He maps a subgroup of the source of tr_{2i-1} to a quotient of the target, and relates the index of this map to the p-adic L-function directly (without using the main theorem of Iwasawa theory).

Note the difference between the case i even and i odd: In the former case, the Euler characteristic of $\mathbb{Q}_p/\mathbb{Z}_p(i)$ gives a a result on the p-adic L-function at $1-i$ which translates into a result for the ζ-function at $1-i$,

because the p-adic L-function approximates the ζ-function at negative integers. In the latter case, the Euler characteristic of $\mathbb{Q}_p/\mathbb{Z}_p(1-i)$ only gives a result for the p-adic L-functions at i.

We give a version for K-theory instead of étale K-theory:

COROLLARY 4.6. *Assume $i > 1$ is an odd integer.*
a) The trace map factors like

$$\mathrm{tr}'_{2i-1} : K_{2i-1}(\mathcal{O}) \otimes \mathbb{Z}_p/tors \to \mathrm{TC}_{2i-1}(\mathcal{O}, \mathbb{Z}_p).$$

b) $S(K, 1-i)$ implies

$$|\operatorname{coker} \mathrm{tr}'_{2i-1}| \cdot |H^2(G_\Sigma, \mathbb{Z}_p(i))| = \prod_{\mathfrak{p}|p} w_{\mathfrak{p},1-i} \cdot |L_p(K, \omega^{1-i}, i)|_p^{-1}.$$

c) If moreover $K/\mathbb{Q}$ is unramified at p and $i \not\equiv 1 \mod p-1$, then tr'_{2i-1} is a map between free $\mathbb{Z}_p$-modules of rank d, and

$$|\operatorname{coker} tr'_{2i-1}| \cdot |H^2(G_\Sigma, \mathbb{Z}_p(i))| = |L_p(K, \omega^{1-i}, i)|_p^{-1}.$$

d) Let $K = \mathbb{Q}$, $i \not\equiv 1 \mod p-1$, and n a positive integer with $n \equiv -i \mod p-1$. If $p | \frac{B_{n+1}}{n+1}$, then $H^2(G_\Sigma, \mathbb{Z}_p(i)) \neq 0$ or tr'_{2i-1} is not surjective.

PROOF. a) By Lemma 4.3 b) and Proposition 4.1

$$K^{\text{ét}}_{2i-1}(\mathcal{O}, \mathbb{Z}_p)_{tors} = H^1(G_\Sigma, \mathbb{Z}_p(i))_{tors} = H^0(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0,$$

so the trace map factors through the torsion free quotient of $K_{2i-1}(\mathcal{O}) \otimes \mathbb{Z}_p$.

b) By Soulé [**22**] the map $K_{2i-1}(\mathcal{O}, \mathbb{Z}_p)/tors \to K^{\text{ét}}_{2i-1}(\mathcal{O}, \mathbb{Z}_p)/tors$ is surjective. Since both groups are free $\mathbb{Z}_p$-modules of the same rank, they must be isomorphic. In other words, $K_{2i-1}(\mathcal{O}, \mathbb{Z}_p)/tors \cong H^1(G_\Sigma, \mathbb{Z}_p(i))$, and the statement is a just a reformulation of the previous theorem.

c) The torsion subgroup of $\mathrm{TC}_{2i-1}(\mathcal{O}; p, \mathbb{Z}_p) \cong \prod_{\mathfrak{p}|p} H^1(K_\mathfrak{p}, \mathbb{Z}_p(i))$ is isomorphic to $\prod_{\mathfrak{p}|p} H^0(K_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$ by Lemma 4.3 c).

d) By [**31**, Thm. 5.11, Cor. 5.13],

$$\begin{aligned} L_p(\mathbb{Q}, \omega^{1-i}, i) = L_p(\mathbb{Q}, \omega^{n+1}, i) &\equiv L_p(\mathbb{Q}, \omega^{n+1}, 1-(n+1)) \\ &\equiv -(1-p^n)\frac{B_{n+1}}{n+1} \equiv -\frac{B_{n+1}}{n+1} \mod p. \end{aligned}$$

□

References

[1] P.BAYER, J.NEUKIRCH, On values of zeta functions and l-adic Euler characteristics. Inv. Math. **50** (1978), 35–64.

[2] K.S.Brown, S.M.Gersten, Algebraic K-theory as generalized sheaf cohomology. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst. Seattle, Wash., 1972), pp. 266–292. Lecture Notes in Math., Vol. **341**, Springer, Berlin, 1973.

[3] J.Coates, S.Lichtenbaum, On l-adic zeta functions. Ann. of Math. (2) **98** (1973), 498–550.

[4] O.Gabber, K-theory of Henselian local rings and Henselian pairs. Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989). Cont. Math. **126** (1992), 59–70.

[5] T.Geisser, p-adic K-theory of Hecke characters of imaginary quadratic fields. Duke Math. J. **86** (1997), 197–238.

[6] T.Geisser, Motivic cohomology, K-theory and topological cyclic homology. To appear in: K-theory handbook.

[7] T.Geisser, L.Hesselholt, Topological Cyclic Homology of Schemes. K-theory, Proc. Symp. Pure Math. AMS **67** (1999), 41–87.

[8] T.Geisser, L.Hesselholt, K-theory and topological cyclic homology of smooth schemes over discrete valuation rings. Trans. Amer. Math. Soc. (to appear)

[9] T.Geisser, L.Hesselholt, The de Rham-Witt complex and p-adic vanishing cycles. Preprint 2003.

[10] T.Geisser, M.Levine, The p-part of K-theory of fields in characteristic p. Inv. Math. **139** (2000), 459–494.

[11] L.Hesselholt, Algebraic K-theory and trace invariants. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 415–425, Higher Ed. Press, Beijing, 2002.

[12] L.Hesselholt, I.Madsen, On the K-theory of finite algebras over Witt vectors of perfect fields. Topology **36** (1997), no. 1, 29–101.

[13] L.Hesselholt, I.Madsen, The K-theory of local fields. To appear in: Annals of Math.

[14] U.Jannsen, Continuous étale cohomology. Math. Ann. **280** (1988), 207–245.

[15] M.Kolster, Nguyen Quang Do, Thong, Syntomic regulators and special values of p-adic L-functions. Invent. Math. **133** (1998), no. 2, 417–447.

[16] R.McCarthy, Relative algebraic K-theory and topological cyclic homology. Acta Math. **179** (1997), 197–222.

[17] J.S.Milne, Arithmetic duality theorems. Perspectives in Mathematics, 1. Academic Press, Inc., Boston, MA, 1986.

[18] J.S.Milne, Etale cohomology. Princeton Math. Series 33.

[19] Y.Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **279** (1989), 241–342.

[20] I.A.Panin, The Hurewicz theorem and K-theory of complete discrete valuation rings. Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 763–775.

[21] P.Schneider, Über gewisse Galoiskohomologiegruppen. Math. Zeit. **168** (1979), 181–205.

[22] C.Soulé, K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. Inv. Math. **55** (1979), 251–295.

[23] C.Soulé, On higher p-adic regulators. Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), pp. 372–401, Lecture Notes in Math., **854**, Springer, Berlin-New York, 1981.

[24] C.Soulé, Operations on etale K-theory. Applications. Algebraic K-theory, Part I (Oberwolfach, 1980), pp. 271–303, Lecture Notes in Math., **966**, Springer, Berlin, 1982.
[25] C.Soulé, p-adic K-theory of elliptic curves. Duke Math. J. **54** (1987), 249–269.
[26] A.Suslin, On the K-theory of local fields. J. Pure Appl. Alg. **34** (1984), 301–318.
[27] R.W.Thomason, Bott stability in algebraic K-theory. Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 389–406, Contemp. Math., **55**, Amer. Math. Soc., Providence, RI, 1986.
[28] R.W.Thomason, Algebraic K-theory and etale cohomology. Ann. Sci. ENS **18** (1985), 437–552.
[29] R.Thomason, T.Trobaugh, Higher algebraic K-theory of schemes and of derived categories. The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., **88**, Birkhäuser Boston, Boston, MA, 1990.
[30] V.Voevodsky, On motivic cohomology with $\mathbb{Z}/l$-coefficients. Preprint 2003.
[31] L.Washington, Introduction to cyclotomic fields. Springer GTM 83 (1982).

University of Southern California, Dep. of Math., KAP 108, 3620 Vermont Av., Los Angeles 90089

E-mail address: `geisser@math.usc.edu`